$\mathcal{N}$-fold Supersymmetry in Quantum Systems with Position-dependent Mass

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Abstract

We formulate the framework of $\mathcal{N}$-fold supersymmetry in one-body quantum mechanical systems with position-dependent mass (PDM). We show that some of the significant properties in the constant-mass case such as the equivalence to weak quasi-solvability also hold in the PDM case. We develop a systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric PDM system. We apply it to obtain type A $\mathcal{N}$-fold supersymmetry in the case of PDM, which is characterized by the so-called type A monomial space. The complete classification and general form of effective potentials for type A $\mathcal{N}$-fold supersymmetry in the PDM case are given.

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I. INTRODUCTION

Recently, quantum systems with position-dependent mass (PDM) have attracted much attention in various research fields of physics such as semiconductors, quantum dots, liquid crystals, and so on. Accordingly, investigations into exact solutions of PDM quantum systems have been carried out increasingly in the last few years. For references, see e.g., Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] and those cited therein. Due to the fact that the position-dependent mass $m(q)$ does not commute with the momentum operator $p = -\mathrm{i}\mathrm{d}/\mathrm{d}q$, ambiguity arises in defining a quantum kinetic operator which is formally Hermitian and reduces to the classical kinetic term $T = p^2/2m(q)$. Hence, the following operator proposed by von Roos [27] has been generally considered:

$$H = -\frac{1}{4}\left(m(q)^{\alpha}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\beta}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\gamma} + m(q)^{\gamma}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\beta}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\alpha}\right) + V(q), \tag{1.1}$$

with the constraint $\alpha+\beta+\gamma = -1$. A different choice of the parameters results in a different correction to the original potential profile $V(q)$, and the above Hamiltonian always has the following form:

$$H = -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + U(q), \tag{1.2}$$

where the *effective* potential $U(q)$ is given by

$$U(q) = V(q) - (\alpha+\gamma)\frac{m''(q)}{4m(q)^2} + (\alpha\gamma+\alpha+\gamma)\frac{m'(q)^2}{2m(q)^3}. \tag{1.3}$$

Hence, the typical investigations into (quasi-)exact solvability of PDM quantum systems consist in finding simultaneously a pair of an effective potential $U(q)$ and a mass function $m(q)$ for which the PDM Hamiltonian (1.2) admits (a number of) exact eigenfunctions in closed form. Up to now, two different methods have been frequently employed, namely, coordinate transformations including point canonical transformations [1, 2, 5, 6, 7, 8, 9, 12, 16, 18, 20, 25], and supersymmetric methods [3, 4, 7, 8, 11, 13, 14, 19, 20, 21, 22, 24, 25]. The latter approaches were also applied to many-body PDM quantum systems [26]. For the methods themselves developed in ordinary quantum systems, see references cited therein.

On the other hand, the framework of $\mathcal{N}$-fold supersymmetry has been rapidly developed in one-body ordinary quantum mechanical systems, especially in the last few years. It was originally proposed as a higher derivative generalization of ordinary supercharges [28]. Later, a significant breakthrough was achieved by the proof of the equivalence between $\mathcal{N}$-fold supersymmetry and (weak) quasi-solvability [29]. Based on the equivalence, a systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric system was developed [30]. Owing to the fact that the framework of $\mathcal{N}$-fold supersymmetry includes all the ordinary supersymmetric methods as its special cases and to its equivalence to weak quasi-solvability, which is less restrictive concept than quasi-exact solvability [31, 32], it provides one of the most powerful methods for investigating one-body quantum mechanical systems which admit analytic solutions in so far the least restrictive sense.

Considering the present situation described above, it is natural to ask whether or not we can formulate $\mathcal{N}$-fold supersymmetry also in PDM quantum systems, especially as a

powerful tool for constructing (quasi-)solvable PDM systems. In this paper, we show that it is indeed possible and that the characteristic properties such as the equivalence to weak quasi-solvability also hold in the PDM case. Furthermore, we generalize the systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric ordinary quantum system in Ref. [30] to the PDM case. As an illustration, we apply it to construct so called type A $\mathcal{N}$-fold supersymmetry [33, 34] in PDM quantum systems.

The paper is organized as follows. In the following section, we review the precise definition of some important concepts in the paper, such as quasi-solvability, to avoid ambiguity. In Section III, we define $\mathcal{N}$-fold supersymmetry in one-body quantum systems with position-dependent mass and discuss a couple of general significant consequences. In Section IV, we develop a systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric PDM quantum system by slightly generalizing the one in the constant-mass case in Ref. [30]. In Section V, we apply the algorithm to construct type A $\mathcal{N}$-fold supersymmetry in the PDM case. In Section VI, we completely classify and present the explicit forms of all the inequivalent type A models with arbitrary position dependence of mass. In the last section, we summarize the results and discuss various future issues.

II. DEFINITION

First of all, we shall give precise definition of some key concepts in the paper. The original ideas can be found in Refs. [29, 35] but we have slightly modified the terms from the view point of the recent advances in this research field. Let H be a linear differential operator of a single variable and $P_{\mathcal{N}}$ be an (at most) $\mathcal{N}$th-order linear differential operator. Then, H is said to be *weakly quasi-solvable* with respect to $P_{\mathcal{N}}$ [34] if it preserves the vector space $\mathcal{V}_{\mathcal{N}}$ defined by

$$\mathcal{V}_{\mathcal{N}} = \ker P_{\mathcal{N}}. \tag{2.1}$$

A linear differential operator H of a single variable is said to be *quasi-solvable* if it preserves a finite-dimensional functional space $\mathcal{V}_{\mathcal{N}}$ whose basis admits an analytic expression in closed form:

$$H\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}}, \qquad \dim \mathcal{V}_{\mathcal{N}} = n(\mathcal{N}) < \infty, \qquad \mathcal{V}_{\mathcal{N}} = \left\langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \right\rangle. \tag{2.2}$$

It is evident that a weakly quasi-solvable operator is also quasi-solvable if linearly independent $\mathcal{N}$ solutions of $P_{\mathcal{N}}\phi = 0$ can be obtained in closed form. An immediate consequence of the above definition of quasi-solvability is that, since we can calculate finite-dimensional matrix elements $\mathsf{H}_{k,l}$ defined by

$$H\phi_k = \sum_{l=1}^{n(\mathcal{N})} \mathsf{H}_{k,l}\phi_l, \qquad k = 1, \dots, n(\mathcal{N}), \tag{2.3}$$

we can diagonalize the operator H and obtain its spectra in the space $\mathcal{V}_{\mathcal{N}}$ with finite algebraic manipulations. However, these calculable spectra and the corresponding vectors of $\mathcal{V}_{\mathcal{N}}$ in general only give *local* solutions of the characteristic equation. This fact naturally leads to the well-known concept of quasi-exact solvability. A quasi-solvable operator H is said to be *quasi-exactly solvable* (on $S \subset \mathbb{R}$ or $\mathbb{C}$) if the invariant space $\mathcal{V}_{\mathcal{N}}$ is a subspace of a

Hilbert space $L^2(S)$ on which the operator H is naturally defined. It is evident that if an operator is not only quasi-solvable but also quasi-exactly solvable, the calculable spectra and the corresponding vectors of $\mathcal{V}_{\mathcal{N}}$ give a part of the *exact* eigenvalues and eigenfunctions of H, respectively. On the other hand, there are cases in which all the eigenfunctions and eigenvalues can be obtained analytically such as the well-known harmonic oscillator. In order to characterize these cases, we first introduce another subclass of quasi-solvability. A quasi-solvable operator H is said to be *solvable* if it preserves an infinite flag of finite-dimensional functional spaces $\mathcal{V}_{\mathcal{N}}$,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{\mathcal{N}} \subset \cdots, \tag{2.4}$$

whose bases admit explicit analytic expressions in closed form, that is,

$$H\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}}, \qquad \dim \mathcal{V}_{\mathcal{N}} = n(\mathcal{N}) < \infty, \qquad \mathcal{V}_{\mathcal{N}} = \left\langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \right\rangle, \tag{2.5}$$

for all $\mathcal{N} = 1, 2, 3, \dots$. A consequence of solvability defined above is that, for an arbitrary natural number $\mathcal{N}$, we can obtain additional $n(\mathcal{N}+1) - n(\mathcal{N})$ local solutions of the characteristic equation for H in $\mathcal{V}_{\mathcal{N}+1}$ with finite algebraic manipulations, based on the knowledge of them in the subspace $\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}+1}$. Hence, arbitrary number of local solutions of the characteristic equation for H can be calculated, in principle, with a well-defined finite algebraic algorithm. As in the case of quasi-solvability, however, they are not necessarily the exact eigenfunctions and eigenvalues of H. A solvable operator H is said to be *exactly solvable* (on $S \subset \mathbb{R}$ or $\mathbb{C}$) if the sequence of the spaces (2.4) in S satisfies,

$$\overline{\mathcal{V}_{\mathcal{N}}(S)} \to L^2(S) \qquad (\mathcal{N} \to \infty). \tag{2.6}$$

This definition precisely fits what is commonly meant by exactly solvable. In the PDM case, it is pointed out [22] that in addition to the square integrability, any eigenfunction ψ of a PDM Hamiltonian (1.2) should satisfy on the boundary ∂S

$$\frac{|\psi(q)|^2}{m(q)^{\frac{1}{2}}} \to 0 \qquad (q \to \partial S), \tag{2.7}$$

in order for the PDM Hamiltonian to be Hermitian. In this paper, however, we do not discuss Hilbert spaces, boundary conditions, normalizability, etc. which strongly rely on a specific choice of $m(q)$. We do not assume Hermiticity of Hamiltonians and thus reality of effective potential $U(q)$ and mass function $m(q)$ either, taking into account possible application to recently developing non-Hermitian quantum theories.

III. $\mathcal{N}$-FOLD SUPERSYMMETRY IN PDM QUANTUM SYSTEMS

To define $\mathcal{N}$-fold supersymmetry in one-body PDM quantum systems, let us first introduce fermionic variables ψ and $\psi^\dagger$ satisfying

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \qquad \{\psi, \psi^\dagger\} = 1. \tag{3.1}$$

In this paper, we consider a super-Hamiltonian

$$\mathbf{H}_{\mathcal{N}} = H_{\mathcal{N}}^{-}\psi\psi^\dagger + H_{\mathcal{N}}^{+}\psi^\dagger\psi, \tag{3.2}$$

where the components $H^{\pm}_{\mathcal{N}}$ are given by PDM Schrödinger operators:

$$H^{\pm}_{\mathcal{N}} = -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + U^{\pm}_{\mathcal{N}}(q). \tag{3.3}$$

For the above system, we define $\mathcal{N}$-fold supercharges $\mathbf{Q}^{\pm}_{\mathcal{N}}$ by

$$\mathbf{Q}^{-}_{\mathcal{N}} = P_{\mathcal{N}}\psi^{\dagger} \equiv P^{-}_{\mathcal{N}}\psi^{\dagger}, \quad \mathbf{Q}^{+}_{\mathcal{N}} = P^{t}_{\mathcal{N}}\psi \equiv P^{+}_{\mathcal{N}}\psi, \tag{3.4}$$

where the operator $P_{\mathcal{N}}$ is an $\mathcal{N}$th-order linear differential operator of the following form:[1]

$$P_{\mathcal{N}} = m(q)^{-\frac{\mathcal{N}}{2}}\frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}q^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} w_k(q)\frac{\mathrm{d}^k}{\mathrm{d}q^k}. \tag{3.5}$$

In Eq. (3.4), the superscript t denotes the formal transposition. A system $\mathbf{H}_{\mathcal{N}}$ is said to be *$\mathcal{N}$-fold supersymmetric* with respect to $\mathbf{Q}^{\pm}_{\mathcal{N}}$ if it commutes with them:

$$\left[\mathbf{Q}^{\pm}_{\mathcal{N}}, \mathbf{H}_{\mathcal{N}}\right] = 0. \tag{3.6}$$

From the above simple generalization of $\mathcal{N}$-fold supersymmetry to PDM systems, we can show that most of the relevant consequences of $\mathcal{N}$-fold supersymmetry in ordinary quantum systems with constant mass also hold in the present case. Let us first prove the equivalence between $\mathcal{N}$-fold supersymmetry and weak quasi-solvability. Suppose a super-Hamiltonian $\mathbf{H}_{\mathcal{N}}$ is $\mathcal{N}$-fold supersymmetric. It is easy to see that the component operators $H^{\pm}_{\mathcal{N}}$ satisfy the following intertwining relation

$$P^{-}_{\mathcal{N}}H^{-}_{\mathcal{N}} = H^{+}_{\mathcal{N}}P^{-}_{\mathcal{N}}, \tag{3.7}$$

and its formal transposition, $P^{+}_{\mathcal{N}}H^{+}_{\mathcal{N}} = H^{-}_{\mathcal{N}}P^{+}_{\mathcal{N}}$, and thus they preserve the vector spaces $\ker P^{\pm}_{\mathcal{N}}$, respectively. Hence, $H^{\pm}_{\mathcal{N}}$ are weakly quasi-solvable. Conversely, let us assume that a PDM Schrödinger operator H is weakly quasi-solvable with respect to an $\mathcal{N}$th-order linear differential operator $P_{\mathcal{N}}$ of the form (3.5). Define a linear operator $G = P_{\mathcal{N}}H - H_1 P_{\mathcal{N}}$ with

$$H_1 = H + m^{\frac{\mathcal{N}-2}{2}}w'_{\mathcal{N}-1} + \frac{\mathcal{N}-1}{2}m^{\frac{\mathcal{N}-4}{2}}m'w_{\mathcal{N}-1} + \frac{\mathcal{N}^2 m''}{4m^2} - \frac{3\mathcal{N}^2(m')^2}{8m^3}. \tag{3.8}$$

From the assumed weak quasi-solvability, $H\ker P_{\mathcal{N}} \subset \ker P_{\mathcal{N}}$, we have

$$\phi \in \ker P_{\mathcal{N}} \quad \Longrightarrow \quad G\phi = P_{\mathcal{N}}H\phi - H_1 P_{\mathcal{N}}\phi = 0. \tag{3.9}$$

On the other hand, it is easy to check that G is of order (at most) $\mathcal{N}-1$ while the dimension of $\ker P_{\mathcal{N}}$ is $\mathcal{N}$. Hence, Eq. (3.9) cannot be hold unless G is a null operator. Therefore, if we put $H^{-}_{\mathcal{N}} = H$, $H^{+}_{\mathcal{N}} = H_1$, $P^{-}_{\mathcal{N}} = P_{\mathcal{N}}$, and $P^{+}_{\mathcal{N}} = P^{t}_{\mathcal{N}}$, they satisfy the intertwining relation (3.7) and thus compose an $\mathcal{N}$-fold supersymmetric system.

Another significant consequence of $\mathcal{N}$-fold supersymmetry in ordinary quantum systems is that the anti-commutator of $\mathcal{N}$-fold supercharges $\mathbf{Q}^{-}_{\mathcal{N}}$ and $\mathbf{Q}^{+}_{\mathcal{N}}$ is a polynomial of degree $\mathcal{N}$ in the super-Hamiltonian $\mathbf{H}_{\mathcal{N}}$, the polynomial being (proportional to) the characteristic

[1] Here we fix the irrelevant overall multiplicative constant factor so that $P_{\mathcal{N}}$ is monic when $m(q) = 1$.

polynomial of the component Hamiltonians $H_{\mathcal{N}}^{\pm}$ restricted in the invariant subspaces $\mathcal{V}_{\mathcal{N}}^{\pm}$ [29, 36]:

$$\{\mathbf{Q}_{\mathcal{N}}^{-}, \mathbf{Q}_{\mathcal{N}}^{+}\} \propto \det \left(H_{\mathcal{N}}^{\pm}\big|_{\mathcal{V}_{\mathcal{N}}^{\pm}} - \mathbf{H}_{\mathcal{N}} \right). \tag{3.10}$$

We can easily prove that the same is also true in the PDM case. Indeed, in the proof of the theorem on SUSY algebras with T symmetry in Ref. [36], we only need to modify the functions $w_{\mathcal{N}}^{\pm}$ in Eq. (6) as $w_{\mathcal{N}}^{\pm}(x) = (\mp 1)^{\mathcal{N}} m(x)^{-\frac{\mathcal{N}}{2}}$ and the definition of the operators $r_l^{\pm}$ in Eq. (19) as

$$r_l^{-} = m(x)^{-\frac{1}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}x} + \chi_l^{-}(x) \right), \quad r_l^{+} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \chi_l^{+}(x) \right) m(x)^{-\frac{1}{2}}, \quad l = 1, \dots, \mathcal{N}, \tag{3.11}$$

for a proof of Eq. (3.10) in the PDM case. In our definition of Hamiltonians (3.3), the proportional constant in Eq. (3.10) is $2^{\mathcal{N}}$.

Other characteristic features such as the almost isospectral property between a pair of Hamiltonians $H_{\mathcal{N}}^{\pm}$ are easily derived in the same way as in the ordinary case (cf. Ref. [29]), and thus we don't repeat a presentation of them in this paper.

IV. ALGORITHM FOR CONSTRUCTING AN $\mathcal{N}$-FOLD SUSY PDM SYSTEM

In Ref. [30], a systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric ordinary quantum mechanical system was developed. In this section, we show that it can be easily generalized to the PDM case. The starting point is an $\mathcal{N}$-dimensional linear functional space

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{-} = \left\langle \tilde{\varphi}_1(z), \dots, \tilde{\varphi}_{\mathcal{N}}(z) \right\rangle, \tag{4.1}$$

and a second-order linear differential operator

$$\tilde{H}_{\mathcal{N}}^{-} = -A(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - B(z)\frac{\mathrm{d}}{\mathrm{d}z} - C(z), \tag{4.2}$$

which leaves $\tilde{\mathcal{V}}_{\mathcal{N}}$ invariant. Let

$$\tilde{P}_{\mathcal{N}}^{-} = g(z) \left(\frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}z^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} \tilde{w}_k(z) \frac{\mathrm{d}^k}{\mathrm{d}z^k} \right) \tag{4.3}$$

denote the most general $\mathcal{N}$th-order linear differential operator with kernel $\tilde{\mathcal{V}}_{\mathcal{N}}$, where the function $g(z)$ is for the time being undetermined. We shall first construct another second-order linear differential operator of the form

$$\tilde{H}_{\mathcal{N}}^{+} = \tilde{H}_{\mathcal{N}}^{-} - \delta C(z), \tag{4.4}$$

satisfying the intertwining relation

$$\tilde{P}_{\mathcal{N}}^{-} \tilde{H}_{\mathcal{N}}^{-} - \tilde{H}_{\mathcal{N}}^{+} \tilde{P}_{\mathcal{N}}^{-} = 0. \tag{4.5}$$

To this end, note that the l.h.s. of Eq. (4.5) is in general a linear differential operator of order $\mathcal{N}+1$. Equating to zero the coefficients of $\partial_z^{\mathcal{N}+1}$ and $\partial_z^{\mathcal{N}}$ in this operator, we obtain the following two equations for the functions $g(z)$ and $\delta C(z)$:

$$\frac{g'}{g} = \frac{\mathcal{N}}{2}\frac{A'}{A}, \tag{4.6}$$

$$\delta C = \frac{\mathcal{N}(\mathcal{N}-2)}{2}\left(A'' - \frac{(A')^2}{2A}\right) + \mathcal{N}\left(B' - \frac{BA'}{2A}\right) - A'\tilde{w}_{\mathcal{N}-1} - 2A\tilde{w}'_{\mathcal{N}-1}. \tag{4.7}$$

When Eqs. (4.6) and (4.7) are satisfied, the l.h.s. of Eq. (4.5) is a linear differential operator of order at most $\mathcal{N}-1$ annihilating the $\mathcal{N}$-dimensional vector space $\tilde{\mathcal{V}}_{\mathcal{N}}$, and hence it vanishes identically.

The last step in our construction consists in applying a change of variable

$$z = z(q) \tag{4.8}$$

and a *gauge* transformation

$$\tilde{H}^{\pm}_{\mathcal{N}} \mapsto \mathrm{e}^{-\mathcal{W}^{-}_{\mathcal{N}}}\tilde{H}^{\pm}_{\mathcal{N}}\mathrm{e}^{\mathcal{W}^{-}_{\mathcal{N}}}\big|_{z=z(q)} \equiv H^{\pm}_{\mathcal{N}}, \tag{4.9}$$

to simultaneously convert $\tilde{H}^{\pm}_{\mathcal{N}}$ into the PDM Schrödinger form (3.3). Note that it is certainly possible since (by construction) $\tilde{H}^{-}_{\mathcal{N}}$ and $\tilde{H}^{+}_{\mathcal{N}}$ differ by a scalar function only. The appropriate change of variable and gauge transformation are determined by

$$z'(q)^2 = 2m(q)A(z), \tag{4.10}$$

$$\frac{\mathrm{d}\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q} = \frac{z''(q)}{2z'(q)} - \frac{m(q)B(z)}{z'(q)} - \frac{m'(q)}{2m(q)}. \tag{4.11}$$

The effective potentials $U^{\pm}_{\mathcal{N}}$ are given by

$$U^{\pm}_{\mathcal{N}}(q) = \frac{1}{2m(q)}\left[\left(\frac{\mathrm{d}\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q}\right)^2 - \frac{\mathrm{d}^2\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q^2} + \frac{m'(q)}{m(q)}\frac{\mathrm{d}\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q}\right] - C^{\pm}(z(q)), \tag{4.12}$$

where $C^{-}(z) = C(z)$ and $C^{+}(z) = C(z)+\delta C(z)$. From the above construction it immediately follows that the system (3.2) and (3.4), with $H^{\pm}_{\mathcal{N}}$ given by Eq. (4.9) and $P_{\mathcal{N}}$ by

$$P_{\mathcal{N}} = \mathrm{e}^{-\mathcal{W}^{-}_{\mathcal{N}}}\tilde{P}^{-}_{\mathcal{N}}\mathrm{e}^{\mathcal{W}^{-}_{\mathcal{N}}}\big|_{z=z(q)}, \tag{4.13}$$

is $\mathcal{N}$-fold supersymmetric. Indeed, intertwining relation (3.7) follows by applying the gauge transformation and change of variable to Eq. (4.5). Furthermore, it is important to note that the form of $P_{\mathcal{N}}$ in Eq. (4.13) is compatible with Eq. (3.5). From Eqs. (4.6) and (4.10) we have

$$\frac{g'(z)}{g(z)} = \frac{\mathcal{N}z''(q)}{z'(q)^2} - \frac{\mathcal{N}m'(q)}{2m(q)z'(q)}. \tag{4.14}$$

Integrating the latter equation we obtain

$$g(z) = m(q)^{-\frac{\mathcal{N}}{2}}z'(q)^{\mathcal{N}}, \tag{4.15}$$

where we take the proportional constant to be 1. Thus, it immediately follows from Eqs. (4.3) and (4.13) that $P_{\mathcal{N}}$ in Eq. (4.13) is indeed of the form (3.5). It is evident from the construction that the Hamiltonian $H_{\mathcal{N}}^{-}$ preserves the kernel of $P_{\mathcal{N}}$ given by

$$\mathcal{V}_{\mathcal{N}}^{-} \equiv \ker P_{\mathcal{N}} = \big\langle \varphi_1(q), \dots, \varphi_{\mathcal{N}}(q) \big\rangle, \tag{4.16}$$

with

$$\varphi_i(q) = \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{-}} \tilde{\varphi}_i(z)\big|_{z=z(q)} \quad (i = 1, \dots, \mathcal{N}). \tag{4.17}$$

We thus call the space $\mathcal{V}_{\mathcal{N}}^{-}$ the *solvable sector* of $H_{\mathcal{N}}^{-}$.

Although the construction of an $\mathcal{N}$-fold supersymmetric PDM system itself has been completed, we can make it entirely symmetric with respect to the partner Hamiltonians $H_{\mathcal{N}}^{-}$ and $H_{\mathcal{N}}^{+}$. For this purpose, note that from the transposition of the intertwining relation (3.7), it follows that $H_{\mathcal{N}}^{+}$ leaves invariant the kernel of the supercharge

$$P_{\mathcal{N}}^{+} = P_{\mathcal{N}}^{t} = \mathrm{e}^{\mathcal{W}_{\mathcal{N}}^{-}} \tilde{P}_{\mathcal{N}}^{t}\, \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{-}}. \tag{4.18}$$

Using the identity $(\partial_z)^t = -z'(q)\partial_z z'(q)^{-1}$ and Eq. (4.3) with (4.15), we can express $P_{\mathcal{N}}^{+}$ as

$$P_{\mathcal{N}}^{+} = \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{+}} \bar{P}_{\mathcal{N}}^{+}\, \mathrm{e}^{\mathcal{W}_{\mathcal{N}}^{+}}, \tag{4.19}$$

where

$$\bar{P}_{\mathcal{N}}^{+} = m(q)^{-\frac{\mathcal{N}}{2}} z'(q)^{\mathcal{N}} \left[\left(-\frac{\mathrm{d}}{\mathrm{d}z} \right)^{\mathcal{N}} + \sum_{k=0}^{\mathcal{N}-1} \left(-\frac{\mathrm{d}}{\mathrm{d}z} \right)^{k} \tilde{w}_k(z) \right], \tag{4.20}$$

and the function $\mathcal{W}_{\mathcal{N}}^{+}$ is given by

$$\mathcal{W}_{\mathcal{N}}^{+} = -\mathcal{W}_{\mathcal{N}}^{-} + (\mathcal{N} - 1) \ln |z'(q)| - \frac{\mathcal{N}}{2} \ln |m(q)|. \tag{4.21}$$

Now, the partner Hamiltonians $H_{\mathcal{N}}^{\pm}$ are expressed in a completely symmetric way as

$$H_{\mathcal{N}}^{\pm} = \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{\pm}} \bar{\tilde{H}}_{\mathcal{N}}^{\pm}\, \mathrm{e}^{\mathcal{W}_{\mathcal{N}}^{\pm}}, \tag{4.22}$$

where the gauged Hamiltonians $\tilde{H}_{\mathcal{N}}^{-}$ and $\bar{H}_{\mathcal{N}}^{+}$ leave the kernels of the gauged $\mathcal{N}$-fold supercharges $\tilde{\mathcal{V}}_{\mathcal{N}}^{-} = \ker \tilde{P}_{\mathcal{N}}^{-}$ and $\bar{\mathcal{V}}_{\mathcal{N}}^{+} = \ker \bar{P}_{\mathcal{N}}^{+}$, respectively. To express $\mathcal{W}_{\mathcal{N}}^{\pm}$ in a symmetric way, we introduce two functions by

$$W(q) = \frac{1}{2} \left(\frac{\mathrm{d}\mathcal{W}_{\mathcal{N}}^{-}}{\mathrm{d}q} - \frac{\mathrm{d}\mathcal{W}_{\mathcal{N}}^{+}}{\mathrm{d}q} \right), \tag{4.23}$$

$$E(q) = \frac{z''(q)}{z'(q)}. \tag{4.24}$$

From Eq. (4.10), its immediate consequence

$$z''(q) = m(q)A'(z) + \frac{m'(q)A(z)}{z'(q)}, \tag{4.25}$$

and Eq. (4.11), the function $W(q)$ is expressed as

$$W(q) = -\frac{m(q)}{z'(q)}\left(\frac{\mathcal{N}-2}{2}A'(z) + B(z)\right) \equiv -\frac{m(q)}{z'(q)}Q(z). \tag{4.26}$$

We then have

$$\begin{aligned}\mathcal{W}_{\mathcal{N}}^{\pm} &= -\frac{\mathcal{N}}{4}\ln|m(q)| + \frac{\mathcal{N}-1}{2}\int \mathrm{d}q\, E(q) \mp \int \mathrm{d}q\, W(q) \\ &= -\frac{1}{4}\ln|m(q)| + \frac{\mathcal{N}-1}{4}\ln|2A(z)| \pm \int \mathrm{d}z\, \frac{m(q)Q(z)}{2A(z)}. \end{aligned} \tag{4.27}$$

The connection between the gauged Hamiltonians $\bar{H}_{\mathcal{N}}^{+}$ and $\tilde{H}_{\mathcal{N}}^{+}$ follows easily from Eqs. (4.9), (4.22), and (4.23) as

$$\bar{H}_{\mathcal{N}}^{+} = \exp\left(-2\int \mathrm{d}q\, W(q)\right)\tilde{H}_{\mathcal{N}}^{+} \exp\left(2\int \mathrm{d}q\, W(q)\right). \tag{4.28}$$

Using Eqs. (4.2), (4.4), (4.7), (4.10), and (4.26), we obtain the following unified formula for the gauged Hamiltonians:

$$\begin{aligned}\bar{\tilde{H}}_{\mathcal{N}}^{\pm} =& -A(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{\mathcal{N}-2}{2}A'(z) \pm Q(z)\right]\frac{\mathrm{d}}{\mathrm{d}z} - C(z) \\ & -(1\pm 1)\left[\frac{\mathcal{N}-1}{2}Q'(z) - \frac{1}{2}A'(z)\tilde{w}_{\mathcal{N}-1}(z) - A(z)\tilde{w}_{\mathcal{N}-1}'(z)\right]. \end{aligned} \tag{4.29}$$

Interestingly, the form of the gauged Hamiltonian $\bar{H}_{\mathcal{N}}^{+}$ as well as $\tilde{H}_{\mathcal{N}}^{-}$ is completely the same as in the case of constant mass, cf. Eq. (2.45) in Ref. [30]. It should also be noted that the mass dependence of the Hamiltonians $H_{\mathcal{N}}^{\pm}$ given by Eq. (4.22) emerges only through the change of variable determined by Eq. (4.10) and the gauge potentials by Eqs. (4.11) and (4.21) since the gauged Hamiltonians (4.29) do not depend on the mass function $m(q)$. As a consequence, each of the spectrum of the Hamiltonians $H_{\mathcal{N}}^{\pm}$ does not depend on the mass function $m(q)$.[2] This kind of spectral independence from the mass function is first pointed out in Ref. [10].

V. TYPE A $\mathcal{N}$-FOLD SUPERSYMMETRY

We shall now construct type A $\mathcal{N}$-fold supersymmetric PDM quantum systems with the aid of the algorithm just developed in the previous section. Type A $\mathcal{N}$-fold supersymmetry [33] is characterized by the so-called type A monomial space [30] preserved by $\tilde{H}_{\mathcal{N}}^{-}$:

$$\tilde{\mathcal{V}}_{\mathcal{N}} = \langle 1, z, \dots, z^{\mathcal{N}-1}\rangle. \tag{5.1}$$

Applying the algorithm with this type A monomial space, we can construct the most general form of type A $\mathcal{N}$-fold supersymmetric PDM quantum systems. Fortunately, we can omit

[2] Some of the eigenvalues can (dis)appear depending on $m(q)$ since normalizability of the corresponding wave functions does rely on it.

the process of constructing gauged Hamiltonians. Noting the fact that the form of gauged Hamiltonians $\bar{\tilde{H}}_{\mathcal{N}}^{\pm}$ is completely the same as in the constant-mass case, we immediately have [34]

$$\begin{aligned}\bar{\tilde{H}}_{\mathcal{N}}^{\pm} = & -A(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{\mathcal{N}-2}{2}A'(z) \pm Q(z)\right]\frac{\mathrm{d}}{\mathrm{d}z} \\ & - \left[\frac{(\mathcal{N}-1)(\mathcal{N}-2)}{12}A''(z) \pm \frac{\mathcal{N}-1}{2}Q'(z) + R\right],\end{aligned} \tag{5.2}$$

where R is a constant, and $Q(z)$ and $A(z)$ must satisfy

$$\frac{\mathrm{d}^3 Q(z)}{\mathrm{d}z^3} = 0 \quad \text{for} \quad \mathcal{N} \geq 2, \tag{5.3}$$

$$\frac{\mathrm{d}^5 A(z)}{\mathrm{d}z^5} = 0 \quad \text{for} \quad \mathcal{N} \geq 3, \tag{5.4}$$

or equivalently,

$$Q(z) = b_2 z^2 + b_1 z + b_0 \qquad \text{for} \quad \mathcal{N} \geq 2, \tag{5.5}$$

$$A(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \quad \text{for} \quad \mathcal{N} \geq 3, \tag{5.6}$$

where b_i and a_i are constants. The most general $\mathcal{N}$th-order linear differential operator of the form (4.3), with $g(z)$ given by Eq. (4.15), which annihilates the type A space is obviously

$$\tilde{P}_{\mathcal{N}}^{-} = m(q)^{-\frac{\mathcal{N}}{2}} z'(q)^{\mathcal{N}} \frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}z^{\mathcal{N}}}. \tag{5.7}$$

Hence, using Eqs. (4.13), (4.24), and (4.27), we obtain the operator $P_{\mathcal{N}}$ for the type A $\mathcal{N}$-fold supercharge

$$P_{\mathcal{N}} = m(q)^{-\frac{\mathcal{N}}{2}} \prod_{k=0}^{\mathcal{N}-1} \left(\frac{\mathrm{d}}{\mathrm{d}q} + W(q) - \frac{\mathcal{N} m'(q)}{4m(q)} + \frac{\mathcal{N}-1-2k}{2} E(q) \right), \tag{5.8}$$

where the products of operators are ordered according to the following definition:

$$\prod_{k=k_0}^{k_1} A_k = A_{k_1} A_{k_1-1} \ldots A_{k_0}. \tag{5.9}$$

The above $P_{\mathcal{N}}$ indeed reduces to the ordinary type A $\mathcal{N}$-fold supercharge when $m(q) = 1$. The pair of type A PDM Hamiltonians $H_{\mathcal{N}}^{\pm}$ can be expressed in terms of the functions $E(q)$, $W(q)$, and $m(q)$ using the following formulas, which follow from Eqs. (4.10) and (4.26):

$$A'(z) = z'(q) \left(\frac{E(q)}{m(q)} - \frac{m'(q)}{2m(q)^2} \right), \tag{5.10}$$

$$A''(z) = \frac{E'(q) + E(q)^2}{m(q)} - \frac{3m'(q)E(q)}{2m(q)^2} - \frac{m(q)m''(q) - 2m'(q)^2}{2m(q)^3}, \tag{5.11}$$

$$Q'(z) = -\frac{W'(q) + E(q)W(q)}{m(q)} + \frac{m'(q)W(q)}{m(q)^2}. \tag{5.12}$$

From Eqs. (4.22), (4.27), (5.2), and the above formulas, we finally obtain

$$
\begin{aligned}
H_{\mathcal{N}}^{\pm} = & -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + \frac{W(q)^2}{2m(q)} - \frac{\mathcal{N}^2-1}{24m(q)}\bigl(2E'(q) - E(q)^2\bigr) \\
& + \frac{\mathcal{N}^2+2}{24}\frac{m''(q)}{m(q)^2} - \frac{5\mathcal{N}^2+16}{96}\frac{m'(q)^2}{m(q)^3} \pm \mathcal{N}\left(\frac{W'(q)}{2m(q)} - \frac{m'(q)W(q)}{4m(q)^2}\right) - R. \qquad (5.13)
\end{aligned}
$$

This is the most general form of type A $\mathcal{N}$-fold supersymmetric PDM Hamiltonians and exactly reduces to the ordinary type A when $m(q) = 1$. It should be also noted that the above formula is consistent with Eq. (3.8). From the form of the type A $\mathcal{N}$-fold supercharge (5.8) we have

$$
w_{\mathcal{N}-1}(q) = \mathcal{N} m(q)^{-\frac{\mathcal{N}}{2}} W(q) - \frac{\mathcal{N}^2}{4} m(q)^{-\frac{\mathcal{N}+2}{2}} m'(q). \qquad (5.14)
$$

With this formula, it is straightforward to check for the type A Hamiltonians that

$$
\begin{aligned}
H_{\mathcal{N}}^{+} - H_{\mathcal{N}}^{-} &= \frac{\mathcal{N}W'}{m} - \frac{\mathcal{N}m'W}{2m^2} \\
&= m^{\frac{\mathcal{N}-2}{2}} w_{\mathcal{N}-1}' + \frac{\mathcal{N}-1}{2} m^{\frac{\mathcal{N}-4}{2}} m' w_{\mathcal{N}-1} + \frac{\mathcal{N}^2 m''}{4m^2} - \frac{3\mathcal{N}^2 m'^2}{8m^3},
\end{aligned}
$$

and thus consistent with Eq. (3.8). Finally, we shall express the two conditions (5.3) and (5.4) for type A $\mathcal{N}$-fold supersymmetry in terms of $E(q)$, $W(q)$, and $m(q)$. Using Eqs. (4.10), (4.24), and (4.26), we obtain

$$
\left(\frac{\mathrm{d}}{\mathrm{d}q} - E\right)\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}}{\mathrm{d}q} + E\right)\frac{W}{m} = 0 \quad \text{for} \quad \mathcal{N} \geq 2, \qquad (5.15)
$$

$$
\left(\frac{\mathrm{d}}{\mathrm{d}q} - 2E\right)\left(\frac{\mathrm{d}}{\mathrm{d}q} - E\right)\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}}{\mathrm{d}q} + E\right)\left(\frac{E}{m} - \frac{m'}{2m^2}\right) = 0 \quad \text{for} \quad \mathcal{N} \geq 3. \qquad (5.16)
$$

In particular, in the case of $\mathcal{N} = 1$ the system composed of $P_{\mathcal{N}}$ and $H_{\mathcal{N}}^{\pm}$ reduces to

$$
P_1 = \frac{1}{m(q)^{\frac{1}{2}}}\left(\frac{\mathrm{d}}{\mathrm{d}q} + W(q) - \frac{m'(q)}{4m(q)}\right), \qquad (5.17)
$$

$$
\begin{aligned}
H_1^{\pm} = & -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + \frac{W(q)^2}{2m(q)} + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \\
& \pm \left(\frac{W'(q)}{2m(q)} - \frac{m'(q)W(q)}{4m(q)^2}\right) - R, \qquad (5.18)
\end{aligned}
$$

with which the super-Hamiltonian $\mathbf{H}_1$ and supercharges $\mathbf{Q}_1^{\pm}$ defined by Eqs. (3.2) and (3.4) satisfy ordinary superalgebra:

$$
\bigl[\mathbf{Q}_1^{\pm}, \mathbf{H}_1\bigr] = 0, \quad \bigl\{\mathbf{Q}_1^{-}, \mathbf{Q}_1^{+}\bigr\} = 2(\mathbf{H}_{\mathcal{N}} + R). \qquad (5.19)
$$

Hence, it exactly reduces to (ordinary) supersymmetric PDM quantum systems. We note that thanks to the assignment of the functions $W(q)$ and $m(q)$ in Eq. (5.17), which is different from the one conventionally employed in the literature, the resulting pair of $H_1^{\pm}$ has completely the symmetric form.

Case	Canonical Form	$f(u)$
I	$1/2$	u
II	$2z$	u^2
III	$2\nu z^2$	$\mathrm{e}^{2\sqrt{\nu}u}$
IV	$2\nu(z^2-1)$	$\cosh 2\sqrt{\nu}u$
V	$2z^3-g_2z/2-g_3/2$	$\wp(u)$

TABLE I: Canonical forms of $A(z)$ and the functions $f(u)$ which characterize the change of variable. The parameters $\nu, g_2, g_3 \in \mathbb{C}$ satisfy $\nu \neq 0$ and $g_2^3 - 27g_3^2 \neq 0$.

From Eqs. (4.20) and (5.7) the operator $\bar{P}_{\mathcal{N}}^{+}$ in the type A case reads

$$\bar{P}_{\mathcal{N}}^{+} = (-1)^{\mathcal{N}} m(q)^{-\frac{\mathcal{N}}{2}} z'(q)^{\mathcal{N}} \frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}z^{\mathcal{N}}}, \tag{5.20}$$

and thus $\bar{\mathcal{V}}_{\mathcal{N}}^{+} = \ker \bar{P}_{\mathcal{N}}^{+}$ is also the type A monomial space (5.1). Hence, the solvable sectors of the type A Hamiltonians $H_{\mathcal{N}}^{\pm}$ are given by

$$\mathcal{V}_{\mathcal{N}}^{\pm} = \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{\pm}} \langle 1, z, \dots, z^{\mathcal{N}-1} \rangle \big|_{z=z(q)}, \tag{5.21}$$

where $\mathcal{W}_{\mathcal{N}}^{\pm}$ are defined in Eq. (4.27).

Another important consequence of the fact that the type A systems with arbitrary PDM are obtained from the same gauged Hamiltonians (5.2) is that we can obtain completely the same results as in the constant mass case if we follow the procedure of generating the generalized Bender–Dunne polynomial (GBDP) systems $\{\pi_n^{[\mathcal{N}]}(E)\}_{n=0}^{\infty}$ [34, 37]. Hence, the anti-commutator of the type A $\mathcal{N}$-fold supercharges in the PDM case is also proportional to the $\mathcal{N}$th critical GBDP in the type A super-Hamiltonian (for details see Ref. [34] and references cited therein):

$$\{\mathbf{Q}_{\mathcal{N}}^{-}, \mathbf{Q}_{\mathcal{N}}^{+}\} = 2^{\mathcal{N}} \pi_{\mathcal{N}}^{[\mathcal{N}]}(\mathbf{H}_{\mathcal{N}}). \tag{5.22}$$

VI. CLASSIFICATION OF THE MODELS

Owing to the fact that the form of the gauged Hamiltonians does not depend on the mass function $m(q)$, we can classify type A $\mathcal{N}$-fold supersymmetric PDM systems completely the same way as in the constant mass case [34]. In the complex classification scheme based on the $GL(2,\mathbb{C})$ invariance, there are five inequivalent models according to different canonical forms of $A(z)$ given in the second column of Table I.

A new feature due to the position dependence of mass emerges through the change of variable. From Eq. (4.10) the change of variable is determined by

$$\pm u(q) \equiv \pm \int \mathrm{d}q \sqrt{m(q)} = \int \frac{\mathrm{d}z}{\sqrt{2A(z)}}. \tag{6.1}$$

Expressing the variable z in terms of u from this formula, $z = f(u)$, we obtain the change of variable as

$$z = z(q) = f(u)\big|_{u=u(q)}. \tag{6.2}$$

The function $f(u)$ in each of the five cases is given in the third column of Table I. When $m(q) = 1$, it is evident that $u(q) = q$ (up to an additive constant), and thus reproduces all the type A models with constant mass in Ref. [34]. The form of the change of variable (6.2) indicates that it is more convenient to express the type A PDM Hamiltonians (5.13) in terms of $f(u)$. To this end, we must first express derivatives of z with respect to q in terms of $f(u)$ and $m(q)$. For instance, the first derivative of $z(q)$ reads

$$z'(q) = f'(u)u'(q) = m(q)^{\frac{1}{2}} f'(u). \tag{6.3}$$

Similarly, we can derive

$$z''(q) = m(q)f''(u) + \frac{m'(q)}{2m(q)^{\frac{1}{2}}} f'(u), \tag{6.4}$$

$$z'''(q) = m(q)^{\frac{3}{2}} f'''(u) + \frac{3}{2} m'(q)f''(u) + \left(\frac{m''(q)}{2m(q)^{\frac{1}{2}}} - \frac{m'(q)^2}{4m(q)^{\frac{3}{2}}} \right) f'(u). \tag{6.5}$$

Using Eqs. (4.24) and (6.3)–(6.5), we obtain,

$$\begin{aligned} 2E'(q) - E(q)^2 &= \frac{2z'''(q)}{z'(q)} - \frac{3z''(q)^2}{z'(q)^2} \\ &= m(q) \left(\frac{2f'''(u)}{f'(u)} - \frac{3f''(u)^2}{f'(u)^2} \right) + \frac{m''(q)}{m(q)} - \frac{5m'(q)^2}{4m(q)^2}. \end{aligned} \tag{6.6}$$

The function $W(q)$ and its derivative are also expressed in terms of $f(u)$ and $m(q)$ with the aid of Eqs. (4.26), (6.3) and (6.4) as

$$W(q) = -\frac{m(q)^{\frac{1}{2}}}{f'(u)} Q(z) \bigg|_{z=f(u)}, \tag{6.7}$$

$$W'(q) = \left[\left(\frac{m(q)f''(u)}{f'(u)^2} - \frac{m'(q)}{2m(q)^{\frac{1}{2}} f'(u)} \right) Q(z) - m(q)Q'(z) \right]_{z=f(u)}. \tag{6.8}$$

Substituting Eqs. (6.6)–(6.8) into Eq. (5.13), we finally have the expression of the type A PDM Hamiltonians in terms of $f(u)$ and $m(q)$ as follows:

$$\begin{aligned} H_{\mathcal{N}}^{\pm} = &- \frac{1}{2m(q)} \frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{m(q)^2} \frac{\mathrm{d}}{\mathrm{d}q} + \left[\frac{Q(z)^2}{2f'(u)^2} - \frac{\mathcal{N}^2 - 1}{24} \left(\frac{2f'''(u)}{f'(u)} - \frac{3f''(u)^2}{f'(u)^2} \right) \right. \\ &\left. + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \pm \frac{\mathcal{N}}{2} \left(\frac{f''(u)}{f'(u)^2} Q(z) - Q'(z) \right) - R \right]_{z=f(u)}. \end{aligned} \tag{6.9}$$

Similarly, the gauge potentials $\mathcal{W}_{\mathcal{N}}^{\pm}$ are expressed as

$$\mathcal{W}_{\mathcal{N}}^{\pm} = -\frac{1}{4} \ln|m(q)| + \frac{\mathcal{N} - 1}{2} \ln|f'(u)| \pm \int \mathrm{d}u\, \frac{Q(f(u))}{f'(u)}. \tag{6.10}$$

Hence, the solvable sectors of the type A Hamiltonians (5.21) reads

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}} f'(u)^{-\frac{\mathcal{N}-1}{2}} \exp\left(\mp \int \mathrm{d}u \frac{Q(f(u))}{f'(u)} \right) \langle 1, f(u), \dots, f(u)^{\mathcal{N}-1} \rangle. \tag{6.11}$$

Observing the above formulas and noting the fact that $u(q) \to q$ (up to an additive constant) as $m(q) \to 1$, we can find a simple procedure to obtain a type A PDM quantum system from a type A constant-mass model. That is, if we denote the pair of potentials, gauge potentials, and solvable sectors of a type A constant-mass model as $V_{\mathcal{N}}^{(0)\pm}(q)$, $\mathcal{W}_{\mathcal{N}}^{(0)\pm}(q)$, and $\mathcal{V}_{\mathcal{N}}^{(0)\pm}[q]$, respectively, those of the corresponding type A PDM model are given by

$$U_{\mathcal{N}}^{\pm}(q) = V_{\mathcal{N}}^{(0)\pm}(u(q)) + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3}, \tag{6.12a}$$

$$\mathcal{W}_{\mathcal{N}}^{\pm}(q) = -\frac{1}{4}\ln|m(q)| + \mathcal{W}_{\mathcal{N}}^{(0)\pm}(u(q)), \quad \mathcal{V}_{\mathcal{N}}^{\pm}[q] = m(q)^{\frac{1}{4}}\mathcal{V}_{\mathcal{N}}^{(0)\pm}[u(q)]. \tag{6.12b}$$

This result is consistent with the one obtained by the point canonical transformation, see e.g., Eqs. (2.7) and (2.8) in Ref. [6], and Eqs. (10), (13) and (14) in Ref. [8]. Advantages of the framework of $\mathcal{N}$-fold supersymmetry are that it can automatically determine the general functional form of the potentials for which we can obtain (a number of) analytic solutions and that, in a good situation like the present type A case, we can completely classify all the possible (quasi-)solvable models which have a specific type of solutions. In addition, it enables us to obtain simultaneously a pair of almost isospectral Hamiltonians.

In what follows, we show the explicit form of effective potentials and solvable sectors in each case classified as in Table I. We note that type A models become not only quasi-solvable but also solvable when the parameters in the potentials satisfy

$$a_4 = a_3 = b_2 = 0. \tag{6.13}$$

Hence, the models in Cases I–IV are solvable when $b_2 = 0$, while the model in Case V is always only quasi-solvable irrespective of the values of the parameters b_i.

A. Case I: $A(z) = 1/2$, $f(u) = u$

Effective potentials:

$$U_{\mathcal{N}}^{\pm}(q) = \frac{1}{2}\left(b_2 u(q)^2 + b_1 u(q) + b_0\right)^2 \mp \mathcal{N} b_2 u(q) + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \mp \frac{\mathcal{N} b_1}{2} - R. \tag{6.14}$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}} \exp\left(\mp\frac{b_2}{3}u(q)^3 \mp \frac{b_1}{2}u(q)^2 \mp b_0 u(q)\right)\langle 1, u(q), \ldots, u(q)^{\mathcal{N}-1}\rangle. \tag{6.15}$$

B. Case II: $A(z) = 2z$, $f(u) = u^2$

Effective potentials:

$$\begin{aligned} U_{\mathcal{N}}^{\pm}(q) = {} & \frac{b_2^2}{8}u(q)^6 + \frac{b_2 b_1}{4}u(q)^4 + \frac{1}{8}(b_1^2 + 2b_0 b_2 \mp 6\mathcal{N} b_2)u(q)^2 \\ & + \frac{(\mathcal{N} - 1 \pm b_0)(\mathcal{N} + 1 \pm b_0)}{8u(q)^2} + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \mp \frac{\mathcal{N} b_1}{4} + \frac{b_0 b_1}{4} - R. \end{aligned} \tag{6.16}$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}} u(q)^{-\frac{\mathcal{N}-1\pm b_0}{2}} \exp\left(\mp\frac{b_2}{8}u(q)^4 \mp \frac{b_1}{4}u(q)^2\right)\langle 1, u(q)^2, \ldots, u(q)^{2(\mathcal{N}-1)}\rangle. \tag{6.17}$$

C. Case III: $A(z) = 2\nu z^2$, $f(u) = \mathrm{e}^{2\sqrt{\nu}u}$

Effective potentials:

$$U_{\mathcal{N}}^{\pm}(q) = \frac{b_2^2}{8\nu}\mathrm{e}^{4\sqrt{\nu}u(q)} + \frac{b_2}{4\nu}(b_1 \mp 2\mathcal{N}\nu)\mathrm{e}^{2\sqrt{\nu}u(q)} + \frac{b_0}{4\nu}(b_1 \pm 2\mathcal{N}\nu)\mathrm{e}^{-2\sqrt{\nu}u(q)} + \frac{b_0^2}{8\nu}\mathrm{e}^{-4\sqrt{\nu}u(q)} \\ + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} + \frac{b_1^2 + 2b_2 b_0}{8\nu} + \frac{\mathcal{N}^2 - 1}{6}\nu - R. \tag{6.18}$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}} \exp\left(\mp\frac{b_2}{4\nu}\mathrm{e}^{2\sqrt{\nu}u(q)} \pm \frac{b_0}{4\nu}\mathrm{e}^{-2\sqrt{\nu}u(q)} - \frac{2(\mathcal{N}-1)\nu \pm b_1}{2\sqrt{\nu}}u(q)\right) \\ \times \left\langle 1, \mathrm{e}^{2\sqrt{\nu}u(q)}, \dots, \mathrm{e}^{2(\mathcal{N}-1)\sqrt{\nu}u(q)}\right\rangle. \tag{6.19}$$

D. Case IV: $A(z) = 2\nu(z^2 - 1)$, $f(u) = \cosh 2\sqrt{\nu}u$

Effective potentials:

$$U_{\mathcal{N}}^{\pm}(q) = \frac{b_2^2}{8\nu}\sinh^2 2\sqrt{\nu}u(q) + \frac{b_2(b_1 \mp 2\mathcal{N}\nu)}{2\nu}\sinh^2\sqrt{\nu}u(q) + \frac{(b_2 + b_0)(b_1 \pm 2\mathcal{N}\nu)}{8\nu\sinh^2\sqrt{\nu}u(q)} \\ + \frac{(b_2 + b_0 - b_1 \mp 2(\mathcal{N}-1)\nu)(b_2 + b_0 - b_1 \mp 2(\mathcal{N}+1)\nu)}{8\nu\sinh^2 2\sqrt{\nu}u(q)} \\ + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \mp \frac{\mathcal{N}b_2}{2} + \frac{2b_2(b_2 + b_0 + b_1) + b_1^2}{8\nu} + \frac{\mathcal{N}^2 - 1}{6}\nu - R. \tag{6.20}$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}}(\sinh 2\sqrt{\nu}u(q))^{-\frac{\mathcal{N}-1}{2}\mp\frac{b_1}{4\nu}}(\tanh\sqrt{\nu}u(q))^{\mp\frac{b_2+b_0}{4\nu}} \exp\left(\mp\frac{b_2}{4\nu}\cosh 2\sqrt{\nu}u(q)\right) \\ \times \left\langle 1, \cosh 2\sqrt{\nu}u(q), \dots, (\cosh 2\sqrt{\nu}u(q))^{\mathcal{N}-1}\right\rangle. \tag{6.21}$$

E. Case V: $A(z) = 2z^3 - g_2 z/2 - g_3/2$, $f(u) = \wp(u)$

Effective potentials:

$$U_{\mathcal{N}}^{\pm}(q) = \sum_{l=1}^{3}\frac{\eta_l^{\pm}}{8H_l^2[\wp(u(q)) - e_l]} + \frac{(\mathcal{N} - 1 \mp b_2)(\mathcal{N} + 1 \mp b_2)}{8}\wp(u(q)) \\ + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3} \pm \frac{\mathcal{N}b_1}{4} + \frac{b_2 b_1}{4} - R, \tag{6.22}$$

where $e_l = \wp(w_l)$ ($l = 1, 2, 3$) are the values of the Weierstrass function at the half of the fundamental periods $2w_l$ and $H_l^2 = 3e_l^2 - g_2/4$. The coupling constants $\eta_l^{\pm}$ are given by

$$\eta_l^{\pm} = -b_2 e_l(b_2 e_l - 2b_1)(2H_l^2 - 5e_l^2) + (b_1^2 + 2b_2 b_0)e_l^2 - 2b_1 b_0 e_l + b_0^2 \\ + (\mathcal{N}^2 - 1)(H_l^4 - 18e_l^2 H_l^2 + 36e_l^4) \mp 2\mathcal{N}\left[(b_2 e_l - b_1)(5H_l^2 - 12e_l^2)e_l - b_0 H_l^2\right]. \tag{6.23}$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{\pm} = m(q)^{\frac{1}{4}} \prod_{l=1}^{3} \bigl|\wp(u(q)) - e_l\bigr|^{-\frac{\mathcal{N}-1}{4} \mp \frac{b_2 e_l^2 - b_1 e_l + b_0}{4H_l^2}} \bigl\langle 1, \wp(u(q)), \dots, \wp(u(q))^{\mathcal{N}-1} \bigr\rangle. \tag{6.24}$$

VII. DISCUSSION AND SUMMARY

In this paper, we have generalized $\mathcal{N}$-fold supersymmetry in ordinary quantum systems to those with position-dependent mass. The significant properties such as the equivalence to weak quasi-solvability also hold in the PDM case. We have developed the general procedure to construct an $\mathcal{N}$-fold supersymmetric PDM system and applied it to obtain the general form of type A $\mathcal{N}$-fold supersymmetry in PDM quantum systems. it turns out that the framework of $\mathcal{N}$-fold supersymmetry is quite powerful also in searching (quasi-)solvable PDM Hamiltonians as well as ordinary ones. In fact, many of the so far constructed (quasi-)solvable PDM Hamiltonians in the literature are realized as special cases of type A $\mathcal{N}$-fold supersymmetry. In addition, we can simultaneously obtain a pair of almost isospectral PDM Hamiltonians in the framework of $\mathcal{N}$-fold supersymmetry.

There are a lot of future issues worthy investigating in this research direction. For example, it would be straightforward to construct other types of $\mathcal{N}$-fold supersymmetry, namely, type B [38] and type C [30] in PDM quantum systems. They are characterized by respectively the type B monomial space

$$\tilde{\mathcal{V}}_{\mathcal{N}} = \bigl\langle 1, z, \dots, z^{\mathcal{N}-2}, z^{\mathcal{N}} \bigr\rangle, \tag{7.1}$$

and by the type C monomial space

$$\tilde{\mathcal{V}}_{\mathcal{N}} = \bigl\langle 1, z, \dots, z^{\mathcal{N}_1 - 1}, z^{\lambda}, z^{\lambda+1}, \dots, z^{\lambda+\mathcal{N}_2-1} \bigr\rangle, \tag{7.2}$$

with $\lambda \in \mathbb{R} \setminus \{-\mathcal{N}_2, -\mathcal{N}_2 + 1, \dots, \mathcal{N}_1\}$. Applying the algorithm in Section IV with above spaces, we will obtain the general form of type B and type C $\mathcal{N}$-fold supersymmetric PDM quantum systems. We expect the same relations as Eqs. (6.12) since they are universal in view of the point canonical transformation.

Investigation of dynamical properties are interesting and important. In contrast to the ordinary constant-mass case, not only the form of potential but also the position dependence of mass affect various aspects of PDM systems. In particular, dynamical $\mathcal{N}$-fold supersymmetry breaking can take place through the nonperturbative effect due to quantum tunneling [39, 40]. Hence, it is quite interesting if we can experimentally observe such a phenomenon in realistic systems such as semiconductors, quantum dots, and so on.

Construction of (quasi-)solvable quantum many-body systems with position-dependent mass is a challenging problem. Recently, we have developed systematic and powerful methods for constructing quasi-solvable differential operators of arbitrary number of variables and applied them to construct quasi-solvable many-body Hamiltonians with constant mass [41, 42, 43]. We expect that the methods would also work well in the PDM case. Up to now, we have found four different types of quasi-solvable differential operators of arbitrary number of variables, namely, type A [41], type C [42], type A′, and type C′ [43]. In the constant-mass case, most of the obtained Hamiltonians are Calogero–Sutherland and Inozemtsev models associated with classical root systems. Hence, we will obtain a set of mass-deformed quantum systems of these types which preserve (quasi-)solvability. In this respect, we anticipate a many-body generalization of the relations (6.12), which relates a (quasi-)solvable potential in PDM systems with that in constant-mass systems.

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